Classifying States of a Finite Markov Chain with Membrane Computing

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Summary

- Markov chains
- Classification of states of a Markov chain
- A P system classifying states
- Conclusions

Markov Chains

- A *Markov chain* is a sequence $\{X_t : t \in N\}$ of random variables verifying the property:

$$P(X_{t+1} = j / X_0 = i_0, X_1 = i_1, \ldots, X_t = i_t) = P(X_{t+1} = j / X_t = i_t).$$

- Range of the random variables: **the state space**.
- Finite Markov chain:only take the discrete values $e_1, \ldots, e_k$.

- The conditional distribution $p_{ij}(t) = P(X_t = e_j / X_{t-1} = e_i)$ characterizes the Markov chain and it is called **transition probability**.
- $P(t) = (p_{ij}(t))_{1 \leq i,j \leq k}$ **transition probability matrix**.
- A finite Markov chain is **time homogeneous** if

$$P(t) = P = p_{ij} = P(X_n = e_j / X_{n-1} = e_i).$$

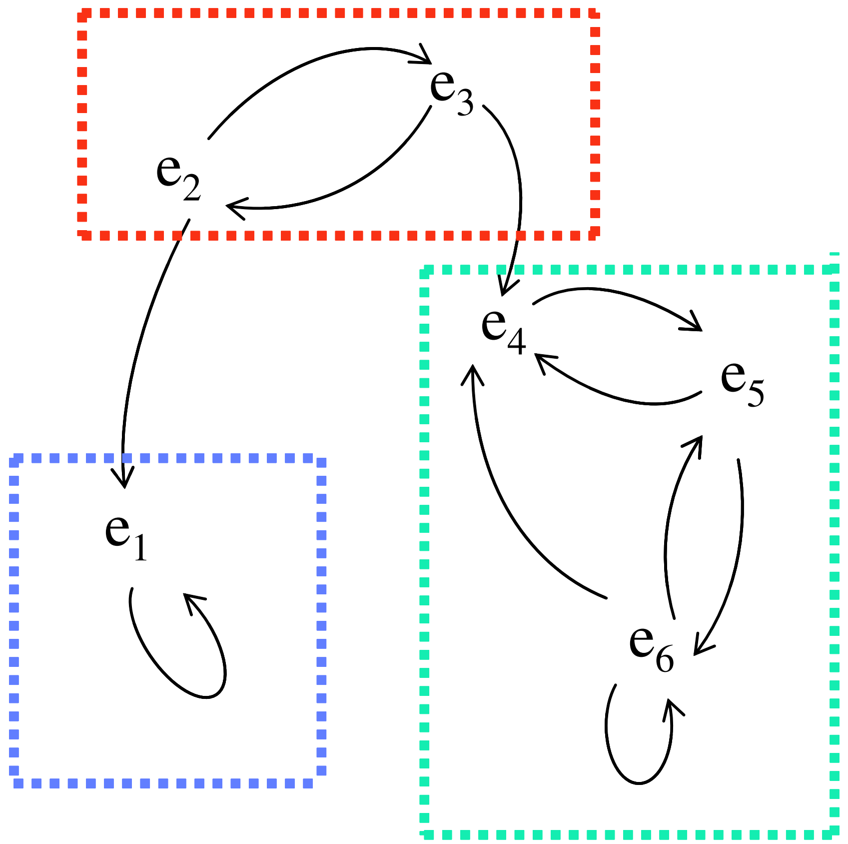
- The homogeneous Markov chain satisfies the conditions

$$p_{ij}^{(1)} = p_{ij},\ p_{ij}^{(n)} = \sum_{r=1}^{k} p_{rj}^{(1)} \cdot p_{ir}^{(n-1)},\ \text{for all } n \geq 2$$

(Kolmogorov–Chapmann equations).

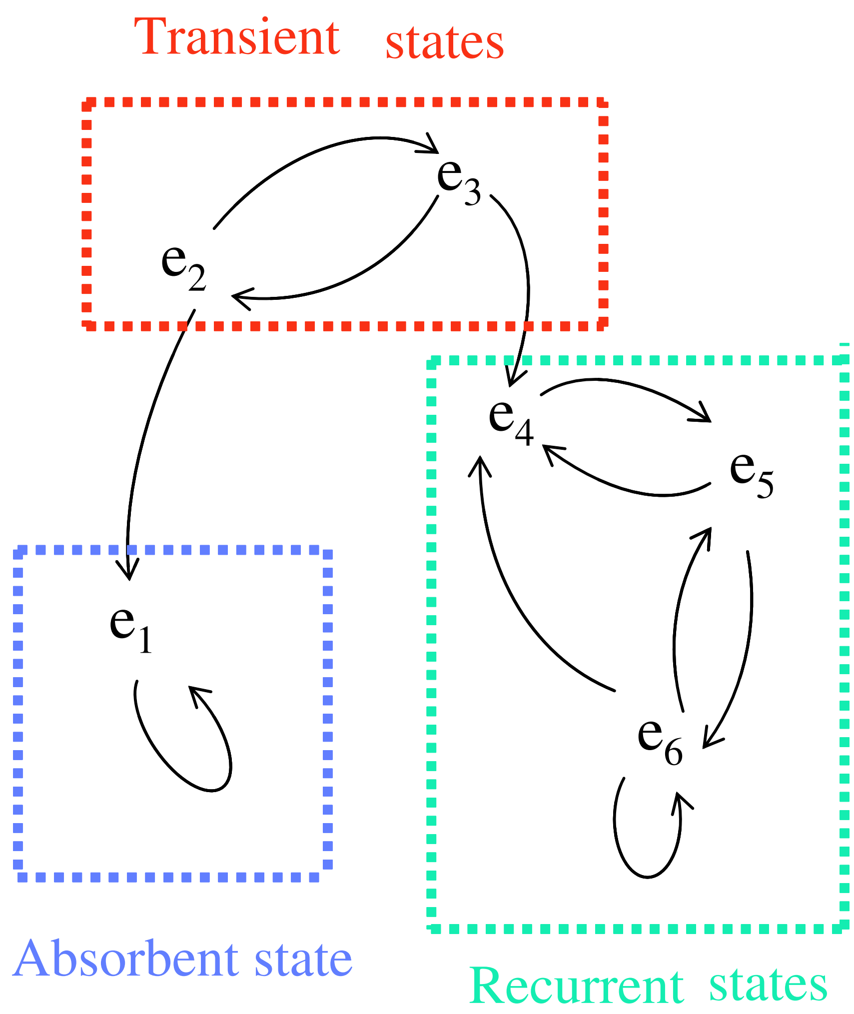
- A finite and homogeneous Markov chain is characterized by the initial probabilities q_0^j and the transition matrix.
- In order to determine the distribution is enough to study the matrix P^n because $q_n = q_0 P^n$.
- The limit of $\{P^n : \; n \in N\}$ allows us to obtain the distribution limit and to know the stationary distribution of the process.

- A state e_j is **accessible** from the state e_i, $(e_i \to e_j)$, if there is an integer $n > 0$ such that $p_{ij}^{(n)} > 0$.
- Two states e_i, e_j are **communicating states,** $(e_i \leftrightarrow e_j)$, if e_i is accessible from e_j and e_j is accessible from e_i.
- The relation of communication is an equivalence relation.



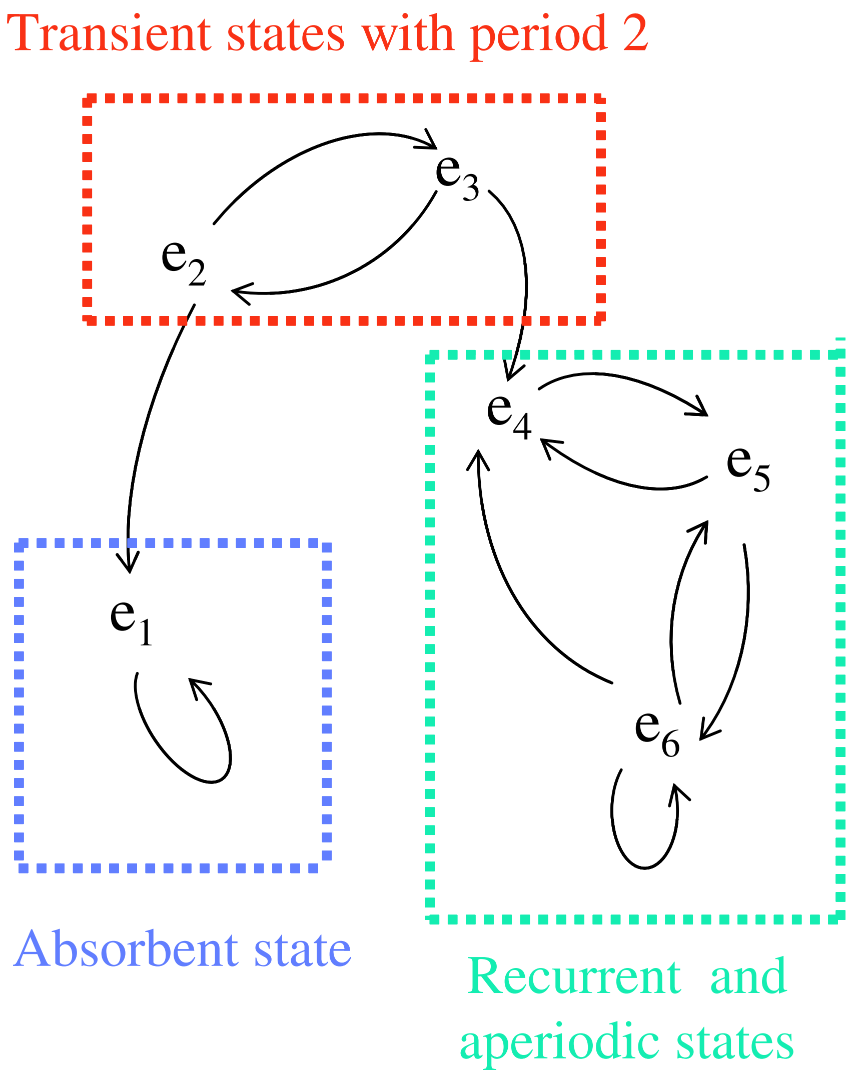
e2
e3
e4
e5
e1
e6

- A state e_i is **recurrent** if for all e_j such that $e_i \to e_j$, then $e_j \to e_i$. On the contrary, e_i is called **transient.**
- If the class of a recurrent state e_i is a singleton, e_i is an **absorbent state**.



Transient states
e3
e2
e4
e5
e1
e6
Absorbent state
Recurrent states

- Given a state e_i such that there exists $n > 0$ and $p_{ii}^{(n)} > 0$, its period is **period** as $d(i) = \text{g.c.d.}\{n \geq 1 \mid p_{ii}^{(n)} > 0\}$.
- All states belonging to the same class have the same period.
- If the period is 1: **aperiodic** class, (otherwise, **periodic** class).


Transient states with period 2
e2
e3
e4
e5
e1
e6
Absorbent state
Recurrent and
aperiodic states

The classification of states allow us:

- Study the asymptotic behavior of stochastic processes.
- Analyze the evolution along the time of a Markov chain.

Theorem 1: There exists a unique stationary distribution iff the set of states contains only one recurrent class.

Theorem 2: There exists a limit distribution iff there is exactly one aperiodic recurrent class.

Goal

Classify the states of a finite Markov chain by using P systems.

Designing a P System

Let $P_k = (p_{ij})_{1\leq i,j\leq k}$ the incidence matrix of the directed graph associated with the Markov chain.

$$\Pi(P_k) = (\Gamma(P_k), \mu(P_k), \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4, R, \rho)$$

- Working alphabet:
  $\Gamma(P_k) = \{a_{ij},\ b_{ij},\ d_{ij},\ t_{ij}:\ 1 \leq i,j \leq k,\} \cup \{c_r:\ 0 \leq r \leq 2k+2\} \cup$
  $\{t_{ijur}:\ 1 \leq i,j,u \leq k,\ 0 \leq r \leq k\} \cup \{\beta_i:\ 0 \leq i \leq \alpha+1\} \cup$
  $\{s_{ijr}:\ 1 \leq i,j \leq k,\ 0 \leq r \leq k\} \cup \{A_{i1},\ \gamma_i:\ 1 \leq i \leq k\} \cup$
  $\{T_{ij},\ R_{ij}:\ 1 \leq i,j \leq k\},\ \alpha = 2k+4+\lceil \lg_2 k\rceil + \frac{(k-1)(k+2)}{2}$

  $s_{ijr} : e_i \to \dots^{r)} \dots \to e_j$ | $t_{ij}\ b_{ij}\ a_{ij} : e_i \to \cdots \to e_j$

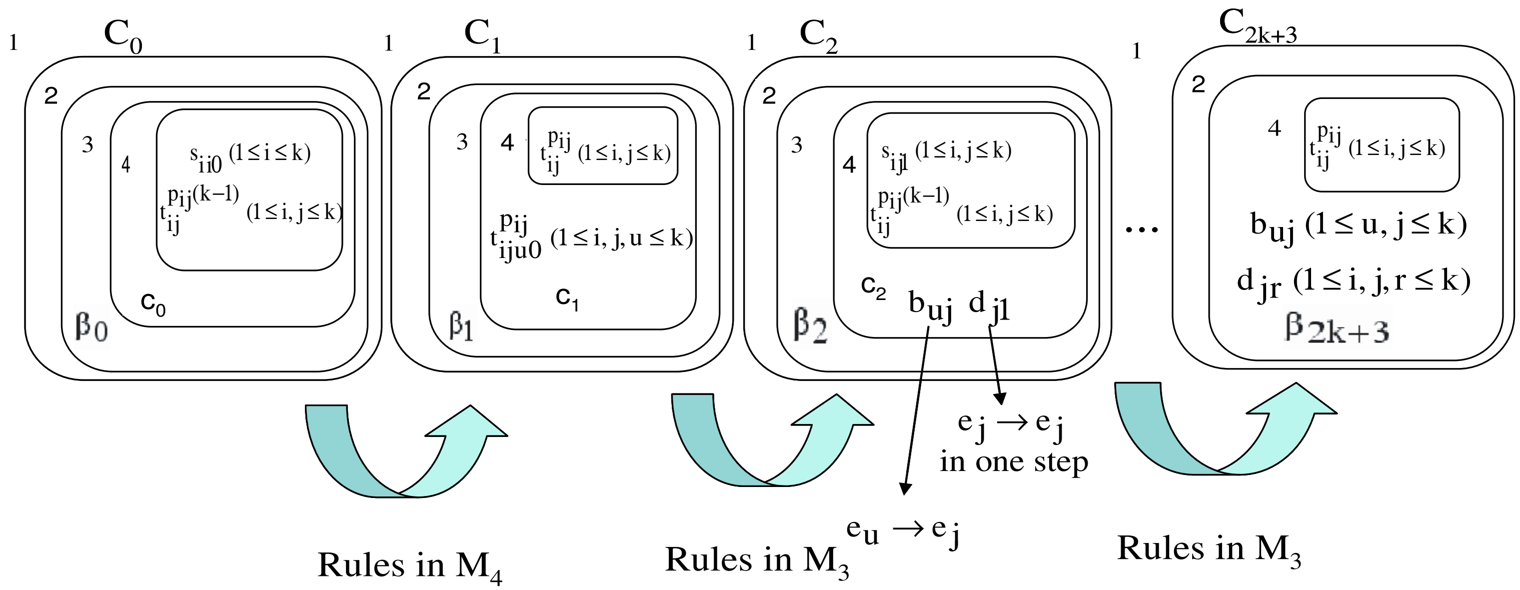
  $t_{ijur} : e_u \to \dots^{r-1)} \dots e_i \to e_j$ | $d_{ij} : e_i \to \dots^{j)} \dots \to e_i$

- Membrane structure: $\mu(P_k) = [_1\ [_2\ [_3\ [_4\ ]_4\ ]_3\ ]_2\ ]_1$

- Initial multisets:
  $\mathcal{M}_1 = \emptyset$; $\mathcal{M}_2 = \{\beta_0\}$; $\mathcal{M}_3 = \{c_0\}$;$\mathcal{M}_4 = \{s_{ii0}\quad t_{ij}^{p_{ij}(k-1)}:\ 1 \leq i,j \leq k\}$

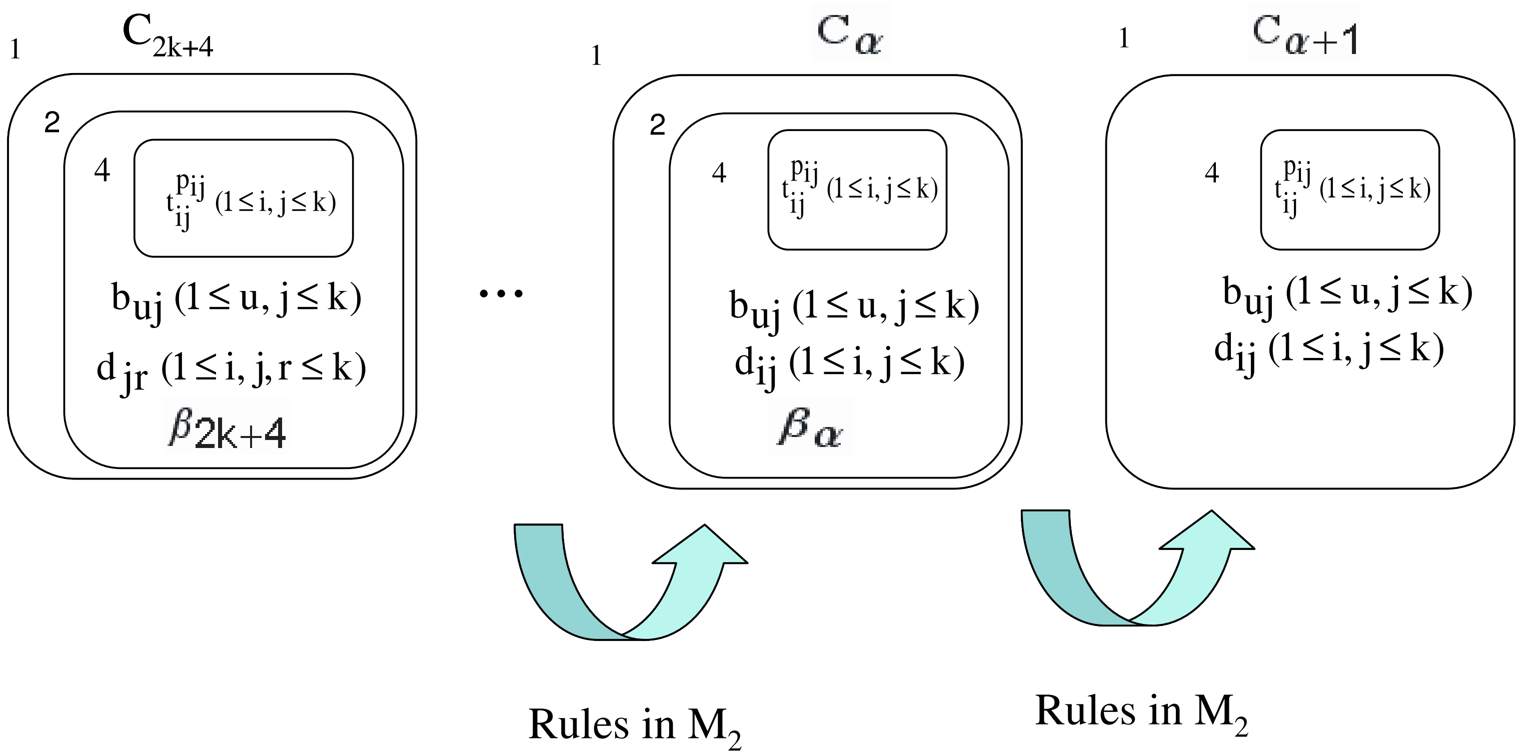
FIRST STAGE: Generating the trajectories of the Markov chain

- Rules in 3: $r_9 = \{t_{ijur} \rightarrow (t_{ij} s_{uj(r+1)}, in_4)\ b_{uj} :\ p_{ij} = 1, u \neq j, 1 \leq i, j, u \leq k, 0 \leq r < k\}$
  $r_{10} = \{t_{ijuk} \rightarrow (t_{ij}, in_4)\ b_{uj} :\ p_{ij} = 1, u \neq j, 1 \leq i, j, u \leq k\}$
  $r_{11} = \{t_{ijjr} \rightarrow (t_{ij}, in_4)\ d_{j(r+1)} :\ p_{ij} = 1, 1 \leq i, j \leq k, 0 \leq r < k\}$
  $r_{12} = \{t_{ijjk} \rightarrow (t_{ij}, in_4) :\ p_{ij} = 1, 1 \leq i, j \leq k\}$
  $r_{13} = \{c_r \rightarrow c_{r+1} :\ 0 \leq r \leq 2k+1\} \cup \{c_{2k+2} \rightarrow \delta\}$
- Rules in 4:
  $r_{14} = \{s_{uir} t_{i1}^{p_{i1}} \ldots t_{ik}^{p_{ik}} \rightarrow (t_{i1ur}^{p_{i1}} \ldots t_{ikur}^{p_{ik}}, out) :\ 1 \leq u, i \leq k,\ 0 \leq r \leq k\}$.



SECOND STAGE: Calculating the period of each state e_i

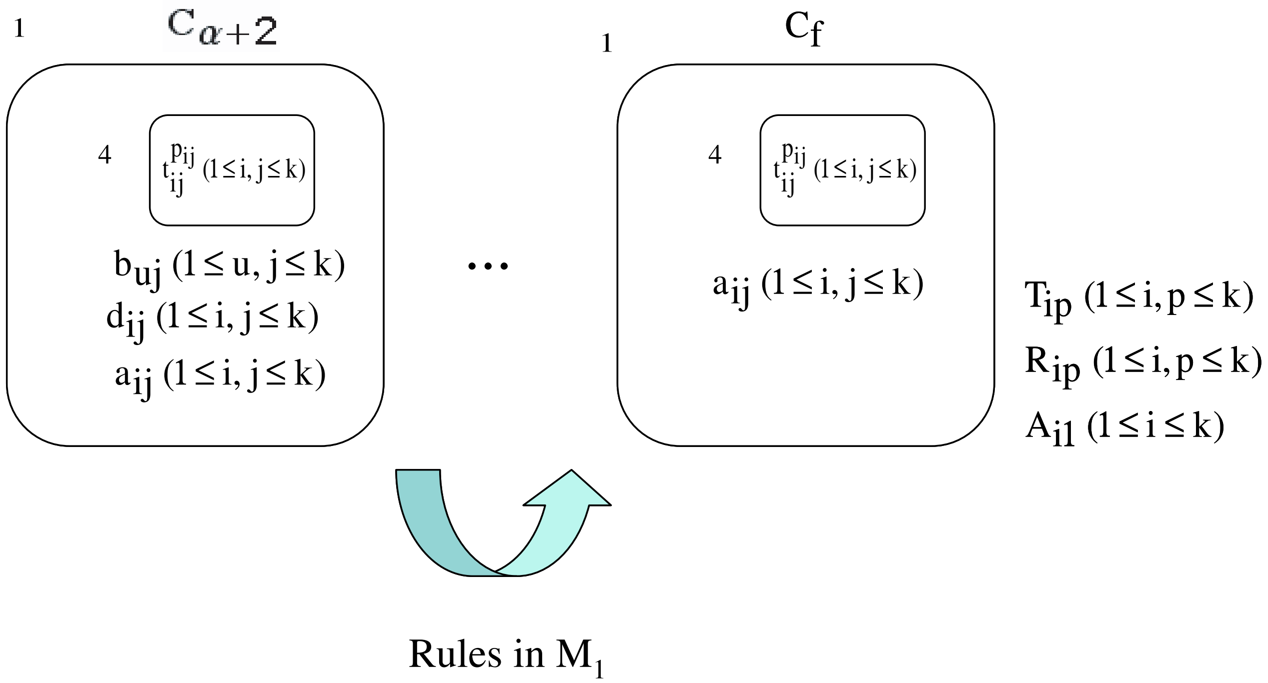
- Rules in 2:
  $r_6 = \{b_{ij}^2 \to b_{ij} : 1 \le i, j \le k\} \cup \{\beta_i \to \beta_{i+1} : 0 \le i \le \alpha\} \cup \{\beta_{\alpha+1} \to \delta\}$.
  $r_7 = \{d_{ij}^2 \to d_{ij} : 1 \le i, j \le k\}$
  $r_8 = \{d_{ij} d_{i(j+l)} \to d_{ij} d_{il} : 1 \le i \le k, 2 \le j + l \le k\}$
  $\{r_7 > r_8\}$



THIRD STAGE: Sending the objects T_{ip}, R_{ip} *and* A_{i1} to the environment

- Rules in 1:

$r_1 = \{b_{ij}b_{ji} \rightarrow a_{ij}a_{ji} : \ 1 \leq i < j \leq k\}$
$r_2 = \{b_{ij} \rightarrow \gamma_i; \quad \gamma_i a_{ij} d_{ip} d_{jp} \rightarrow (T_{ip}T_{jp}, out) : \ 1 \leq i, j, p \leq k\}$
$r_3 = \{\gamma_i d_{ip} \rightarrow (T_{ip}, out) : \ 1 \leq i, j, p \leq k\}$
$r_4 = \{a_{ij} d_{ip} \rightarrow (R_{ip}, out) : \ 1 \leq i, j, p \leq k\}$ $r_5 = \{d_{i1} \rightarrow (A_{i1}, out) : \ 1 \leq i \leq k\}$
$\{r_1 > r_2 > r_3 > r_4 > r_5\}$



Formal verification

Theorem 1 Let C_f be the final configuration of the computation $\mathcal{C}$ of the system $\Pi(P_k)$. Then:

(a) The state e_i is transient with period p if and only if $T_{ip} \in C_f(env)$.

(b) The state e_i is recurrent (and not absorbent) with period p if and only if $R_{ip} \in C_f(env)$.

(a) The state e_i is absorbent (with period 1) if and only if $A_{i1} \in C_f(env)$.

Proposition 1 For each i, j such that $1 \leq i, j \leq k$ we have the following:

(1) If $i \neq j$, then the following assertions are equivalent:

  (a) There exists a path from e_i to e_j.

  (b) The object b_{ij} belongs to $C_{2k+2}(3)$.

  (c) The object b_{ij} belongs to $C_{2k+3}(2)$.

(2) The following conditions are equivalent

  (a) There exists a path from e_i to e_i with length j.

  (b) The object d_{ij} belongs to $C_{2k+2}(3)$.

  (c) The object d_{ij} belongs to $C_{2k+3}(2)$.

Proposition 2 If $\alpha = 2k + 4 + \lceil lg_2 k \rceil + (k - 1)(k + 2)/2$, then:

$$\begin{aligned} C_{\alpha+1}(2) = \quad & \{b_{ij} : \ 1 \leq i, j \leq k, i \neq j, \text{ there is a path from } e_i \text{ to } e_j\} \cup \\ & \{d_{ip} : \ 1 \leq i, p \leq k, \ p \text{ is the period of the state } e_i\} \cup \\ & \{\beta_{\alpha+1}\}. \end{aligned}$$

Conclusions

- We give a solution of the problem of classification of a Finite Markov Chain in the framework of Membrane Computing.
- The solution is:
  - **Semi–uniform**
  - **Efficient**

- The amount of resources initially required to construct the system is polynomial in the order of the Markov chain.

- It provides a new example of a formal verification in computing model oriented to machines following a specific methodology.